On greedy algorithms with respect to generalized Walsh system

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In this paper we proof that there exists a function $f(x)$ belongs to $L^1[0,1]$ such that a greedy algorithm with regard to generalized Walsh system does not converge to $f(x)$ in $L^1[0,1]$ norm, i.e. the generalized Walsh system is not a quasi-greedy basis in its linear span $L^1[0,1]$.

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1 Introduction

In this paper we consider a question of convergence of greedy algorithm with regard to generalized Walsh system in $L^1[0,1]$ norm.

Let X be a Banach space with a norm $||\cdot|| = ||\cdot||_X$ and a basis $\Phi = \{\phi_k\}_{k=1}^{\infty}$, $||\phi_k||_X = 1$, $k = 1, 2, ..$.

Denote by Σ_m the collection of all functions in X which can be expressed as a linear combination of at most m- functions of Φ. Thus each function $g \in \Sigma_m$ can be written in the form

$$g = \sum_{s\in\Lambda} a_s \phi_s, \quad \#\Lambda \le m.$$

For a function $f \in X$ we define its approximate error by

$$\sigma_m(f,\phi) = \inf_{g\in\Sigma_m} ||f-g||_X\,, \quad m = 1,2,...$$

and we consider the expansion

$$f = \sum_{k=1}^{\infty} a_k(f)\phi_k \quad .$$

Definition 1. Let an element $f \in X$ be given. Then the m-th greedy approximant of the function f with regard to the basis Φ is given by

$$G_m(f,\phi) = \sum_{k\in\Lambda} a_k(f)\phi_k, \tag{1}$$

where $\Lambda \subset \{1,2,...\}$ is a set of cardinality m such that

$$|a_n(f)| \geq |a_k(f)|, \quad n \in \Lambda, \quad k \notin \Lambda, \tag{2}$$

We'll say that the greedy approximant of $f(t) \in L^p_{[0,1]}$, $p \geq 0$ converges with regard to the basis Φ , if the sequence $G_m(x,f)$ converges to $f(t)$ in L^p norm. This new and very important direction invaded many mathematician's attention (see [1]-[10]).

Definition 2. We call a basis Φ greedy basis if for every $f \in X$ there exists a subset $\Lambda \subset \{1,2,...\}$ of cardinality m, such that

$$||f - G_m(f,\Phi)||_X \leq C \cdot \sigma_m(f,\Phi)$$

where a constant $C = C(X,\Phi)$ independent of f and m.

In [4] it is proved that each basis Φ which is L_p -equivalent to the Haar basis H is Greedy basis for $L_p(0,1)$, $1 < p < \infty$.

Definition 3. We say that a basis Φ is Quasi-Greedy basis if there exists a constant C such that for every $f \in X$ and any finite set of indices Λ, having the property

$$\min_{k\in\Lambda} |a_k(f)| \geq \max_{n\notin\Lambda} |a_k(f)|$$

we have

$$||S_\Lambda(f,\Phi)||_X = \left|\left|\sum_{k\in\Lambda} a_k(f)\phi_k\right|\right|_X \leq C \cdot ||f||_X.$$

In [5] it is proved that a basis Ψ is quasi-greedy if and only if the sequence $\{G_m(f)\}$ converges to f, for all $f \in X$. Not that the trigonometric and Walsh system are not a quasi-greedy basis for L^p if $1 < p < \infty$ (see [7] and [8]).

2 Definition and properties of generalized Walsh system

Let a denote a fixed integer, $a \geq 2$ and put $\omega_a = e^{\frac{2\pi i}{a}}$.

Now we will give the definitions of Rademacher and generalized Walsh systems (see [12]).

Definition 4. The Rademacher system of order a is defined by

$$\varphi_0(x) = \omega_a^k \quad if \quad x \in \left[\frac{k}{a}, \frac{k+1}{a}\right), \quad k = 0, 1, ..., a-1,$$

and for $n \geq 0$

$$\varphi_n(x+1) = \varphi_n(x) = \varphi_0(a^n x). \tag{3}$$

Definition 5. The generalized Walsh system of order a is defined by

$$\psi_0(x) = 1,$$

and if $n = \alpha_{n_1} a^{n_1} + ... + \alpha_{n_s} a^{n_s}$ where $n_1 > ... > n_s$, then

$$\psi_n(x) = \varphi_{n_1}^{\alpha_{n_1}}(x) \cdot ... \cdot \varphi_{n_s}^{\alpha_{n_s}}(x). \tag{4}$$

Let $\Psi_a = \{\psi_n(x)\}_{n=0}^{\infty}$ denote the generalized Walsh system of order a. Not that Ψ_2 is the classical Walsh system.

Remark. The generalized Walsh system Ψ_a, $a \geq 2$ is a complete orthonormal system in $L^2[0,1)$ (see [12]).

The basic properties of the generalized Walsh system of order a are obtained by R. Paley , H.E.Chrestenson, J. Fine, N. Vilenkin and others (see [11]- [16]).

Define

$$I_{n,k} = I_{n,k}(a) = \left[\frac{k}{a^n}, \frac{k+1}{a^n}\right), \quad k = 0, ..., a^n - 1, \quad n = 1, 2, ...$$

If $\varphi_n(x)$ is the nth Rademacher function of order a, then from Definition 4 it follows

$$\varphi_n(x) = \omega_a^k = e^{\frac{2\pi i \cdot k}{a}}, \quad x \in I_{n+1,k}. \tag{5}$$

Note some properties of generalized Walsh system:

Property 1. From definition 5 we have

$$\psi_{a^k+j}(x) = \varphi_k(x) \cdot \psi_j(x), \quad if \quad 0 \leq j \leq a^k - 1. \tag{6}$$

Denote by

$$D_n(t) = \sum_{k=0}^{n-1} \psi_k(t), \tag{7}$$

the Dirichlet kernel by generalized Walsh system.

Property 2. The Dirichlet kernel has the following properties (see [12])

$$D_{a^n}(t) = \begin{cases} a^n, \quad x \in I_{n,0} = [0, \frac{1}{a^n}); \\ \\ 0, \quad x \in [\frac{1}{a^n}, 1). \end{cases} \tag{8}$$

Property 3. If $n = a^k + m$, $0 \leq m < k$ and consequently by (6) - (8) we have

$$D_n(t) = D_{a^k}(t) + \varphi_k(t) \cdot D_m(t), \quad t \in [0, 1). \tag{9}$$

Property 4. For any natural number m and any $t \in (0, 1)$ the following is true

$$|D_m(t)| \leq m. \tag{10}$$

3 A Basic Lemma

Denote by

$$L_k = \int_0^1 |D_k(t)|dt$$

the kth Lebesgue constant of the generalized Walsh system $\{\Psi_a\}$.

In [12] it is proved that the Lebesgue constant satisfy $L_k = O(\log_a k)$ where O depends upon a. Next Lemma shows that there exists a sequence of natural numbers $\{n_k\}$ so that the sequence L_{n_k} has the same order of growth as $\log_a n_k$. Namely the following is true:

Lemma . There exists a sequence of natural numbers $\{n_k\}_{k=1}^{\infty}$ of the form

$$n_{2s} = \sum_{i=0}^{s} a^{2i} : \quad n_{2s+1} = \sum_{i=0}^{s} a^{2i+1}, \quad s = 0, 1, 2, ..., \tag{11}$$

such that

$$a^k \le n_k < a^{k+1},$$

$$L_{n_k} = \int_0^1 |D_{n_k}(t)|dt > \frac{1}{a} \cdot \left(\frac{k}{2} + 1\right) > \frac{1}{2a} \cdot \log_a n_k, \quad k \ge 1. \tag{12}$$

Proof. Note that

$$n_{2s} = \frac{a^{2s+2} - 1}{a^2 - 1} < \frac{a^2}{a^2 - 1} \cdot a^{2s},$$

$$n_{2s+1} = \frac{a^{2s+3} - a}{a^2 - 1} = a \cdot \frac{a^{2s+2} - 1}{a^2 - 1} < \frac{a^2}{a^2 - 1} \cdot a^{2s+1}$$

i.e.

$$n_k < \frac{a^2}{a^2 - 1} \cdot a^k < a^{k+1}, \quad k = 0, 1, 2, ... \tag{13}$$

From this and (10) we have

$$|D_{n_k}(t)| < \frac{a^2}{a^2 - 1} \cdot a^k, \quad t \in [0, 1), \quad k = 0, 1, 2, ... \tag{14}$$

First we'll prove that

$$\int_{\frac{1}{a^{k+2}}}^{1} |D_{n_k}(t)|dt > \frac{1}{a} \cdot \left(\frac{k}{2} + 1\right) \tag{15}$$

Let $k = 2s$, then we have to prove

$$\int_{\frac{1}{a^{2s+2}}}^{1} |D_{n_{2s}}(t)|dt > \frac{1}{a} \cdot (s+1)\,, \quad s = 0, 1, 2, ... \tag{16}$$

By Definition 5 and (7) for $s = 0$ we have

$$\int_{\frac{1}{a^2}}^{1} |D_1(t)|dt = \int_{\frac{1}{a^2}}^{1} |\psi_0(t)|dt = 1 - \frac{1}{a^2} > \frac{1}{a} \quad .$$

Now assume that for some $s-1$ the inequality (16) holds, i.e.

$$\int_{\frac{1}{a^{2s}}}^{1} |D_{n_{2(s-1)}}(t)|dt > \frac{1}{a} \cdot s. \tag{17}$$

By (8) and (14) we get

$$D_{a^{2s}}(t) = a^{2s}, \quad if \quad t \in I_{2s,0}. \tag{18}$$

$$|D_{n_{2(s-1)}}(t)| < \frac{a^2}{a^2-1} \cdot a^{2(s-1)} = \frac{a^{2s}}{a^2-1}. \tag{19}$$

From (11) it follows that

$$n_{2s} = a^{2s} + n_{2(s-1)} \tag{20}$$

and consequently by (9), (17) and (18) for $t \in I_{2s,0}$ we have

$$|D_{n_{2s}}(t)| = |D_{a^{2s}}(t) + \varphi_{2s}(t) \cdot D_{n_{2(s-1)}}(t)| \geq$$

$$|D_{a^{2s}}(t)| - |D_{n_{2(s-1)}}(t)| > \frac{a^2-2}{a^2-1} \cdot a^{2s}.$$

Hence, taking into account that $\left(\frac{1}{a^{2s+2}}, \frac{1}{a^{2s}}\right) \subset I_{2s,0} = \left[0, \frac{1}{a^{2s}}\right)$, we obtain

$$\int_{\frac{1}{a^{2s+2}}}^{\frac{1}{a^{2s}}} |D_{n_{2s}}(t)|dt > \frac{a^2-2}{a^2-1} \cdot a^{2s} \cdot \left(\frac{1}{a^{2s}} - \frac{1}{a^{2s+2}}\right) =$$

$$\frac{a^2-2}{a^2-1} \cdot \frac{a^2-1}{a^{2s+2}} = \frac{a^2-2}{a^2} \geq \frac{1}{a}. \tag{21}$$

From (8), (9), (17) and (20) follows

$$\int_{\frac{1}{a^{2s}}}^{1} |D_{n_{2s}}(t)|dt > \int_{\frac{1}{a^{2s}}}^{1} |D_{n_{2(s-1)}}(t)|dt > \frac{1}{a} \cdot s.$$

Hence and from (21) we conclude

$$\int_{\frac{1}{a^{2s+2}}}^{1} |D_{n_{2s}}(t)|dt =$$

$$\int_{\frac{1}{a^{2s+2}}}^{\frac{1}{a^{2s}}} |D_{n_{2s}}(t)|dt + \int_{\frac{1}{a^{2s}}}^{1} |D_{n_{2s}}(t)|dt >$$

$$\frac{1}{a} + \frac{1}{a} \cdot s > \frac{1}{a} \cdot (s+1)\,, \quad s = 0, 1, 2, ... \tag{22}$$

In a case $k = 2s+1$ $(s = 0, 1, ...)$ we have to prove

$$\int_{\frac{1}{a^{2s+3}}}^{1} |D_{n_{2s+1}}(t)|dt > \frac{1}{a} \cdot \left(s + \frac{1}{2} + 1\right).$$

For $s = 0$ this inequality holds because in this case $n_{2s+1} = n_1 = a$ and

$$\int_{\frac{1}{a^3}}^{1} |D_a(t)|dt = \int_{\frac{1}{a^3}}^{\frac{1}{a}} a dt = a \cdot \left(\frac{1}{a} - \frac{1}{a^3}\right) =$$

$$\frac{1}{a} \cdot \left(a - \frac{1}{a}\right) \geq \frac{1}{a} \cdot \frac{3}{2}.$$

The next reasonings are similar to a case when $k = 2s$.

Since $a^k \leq n_k < a^{k+1}$ then $\log_a n_k < k+1$ and consequently

$$L_{n_k} = \int_0^1 |D_{n_k}(t)|dt > \int_{\frac{1}{a^{k+2}}}^{1} |D_{n_k}(t)|dt > \frac{1}{a} \cdot \left(\frac{k}{2} + 1\right) >$$

$$\frac{1}{2a} \cdot \left(\frac{k}{2} + 1\right) > \frac{1}{2a} \cdot \log_a n_k.$$

Completing the proof.

4 The Main Theorem and It's Proof.

In [10] we proved the following theorem:

Theorem 1. Let a sequence $\{M_n\}_{n=1}^{\infty}$ be given so that

$$\lim_{k\to\infty}(M_{2k}-M_{2k-1})=+\infty.$$

Then the Walsh subsystem

$$\{W_{n_k}(x)\}_{k=1}^{\infty}=\{W_m(x): \quad M_{2s-1}\le m\le M_{2s}, \quad s=1,2,...\} \tag{1}$$

is not a quasi-greedy basis in its linear span in $L^1[0,1]$.

Let $\Psi_a=\{\psi_n(x)\}_{n=0}^{\infty}$ denote the generalized Walsh system of order a.

From Corollary 2.3 (see [8]) it follows that generalized Walsh system is not a quasi-greedy basis for $L^p[0,1]$ if $1<p<\infty$.

In this paper we prove the following theorem.

Theorem 2. There exists a function $f(x)$ belongs to $L^1[0,1)$ such that the approximate $G_n(f,\Psi_a)$ with regard to the generalized Walsh system does not converge to $f(x)$ by $L^1[0,1)$ norm, i.e. the generalized Walsh system is not a quasi-greedy basis in its linear span in L^1.

Proof. Let $a\ge 2$ denote a fixed integer. For any natural k we set

$$f_k(x)=\sum_{i=a^{(k-1)^2}}^{a^{k^2}-1}\left(\frac{1}{k^2}+2^{-i}\right)\cdot\psi_i(x). \tag{23}$$

It is easy to see that the Fourier coefficients by generalized Walsh system of the function $f_k(x)$ are defined as follows

$$C_i^{(k)}=\frac{1}{k^2}+2^{-i} \quad if \quad a^{(k-1)^2}\le i<a^{k^2}. \tag{24}$$

Now we consider the following function

$$f(x)=\sum_{i=1}^{\infty}C_i\psi_i(x)=\sum_{k=1}^{\infty}f_k(x)=$$

$$= \sum_{k=1}^{\infty} \left[\sum_{i=a^{(k-1)^2}}^{a^{k^2}-1} \left(\frac{1}{k^2} + 2^{-i} \right) \cdot \psi_i(x) \right], \tag{25}$$

where

$$C_i = C_i^{(k)} \quad if \quad a^{(k-1)^2} \leq i < a^{k^2}. \tag{26}$$

Now we will show that $f(x) \in L^1[0,1]$. For this we represent the function $f(x)$ in the following way:

$$f(x) = g(x) + h(x), \tag{27}$$

where

$$g(x) = \sum_{k=1}^{\infty} \frac{1}{k^2} \left[\sum_{i=a^{(k-1)^2}}^{a^{k^2}-1} \psi_i(x) \right] = \sum_{k=1}^{\infty} \frac{1}{k^2} \left[D_{a^{k^2}}(x) - D_{a^{(k-1)^2}}(x) \right],$$

$$h(x) = \sum_{k=1}^{\infty} \left[\sum_{i=a^{(k-1)^2}}^{a^{k^2}-1} 2^{-i} \cdot \psi_i(x) \right] = \sum_{j=1}^{\infty} 2^{-j} \cdot \psi_j(x).$$

For the function $g(x)$ and from (8) and definition 5 we have

$$\int_0^1 |g(x)| dx \leq 2 \cdot \sum_{k=1}^{\infty} \frac{1}{k^2} < \infty$$

which means $g(x) \in L^1[0,1]$.

Analogously

$$\int_0^1 |h(x)| dx \leq \sum_{j=1}^{\infty} \frac{1}{2^j} < \infty$$

i.e. $h(x) \in L^1[0,1]$. Hence and from (27) it follows that $f(x) \in L^1[0,1]$.

For any natural k we choose numbers i, j so that

$$a^{(k-1)^2} \leq i < a^{k^2} \leq j < a^{(k+1)^2}.$$

Then

$$\frac{1}{(k+1)^2} + 2^{-j} < \frac{1}{k^2} + 2^{-i},$$

and from (24) we have $C_j^{(k+1)}(f) < C_i^{(k)}(f)$.

Analogously for any number i, $a^{(k-1)^2} \le i < a^{k^2}$, we have

$$\frac{1}{k^2} + 2^{-(i+1)} < \frac{1}{k^2} + 2^{-i},$$

i.e. $C_{i+1}^{(k)}(f) < C_i^{(k)}(f)$.

Thus for any natural numbers i we get

$$C_{i+1}(f) < C_i(f).$$

On the other hand if $i \to \infty$ then $k \to \infty$ (see (24)). Then from (24) and (26) we get $C_i(f) \searrow 0$.

For any numbers m_k so that

$$a^{(k-1)^2} + m_k < a^{k^2}, \tag{28}$$

by (24) - (26) and Definition 1 we have

$$G_{a^{(k-1)^2}+m_k}(f, \Psi_a) - G_{a^{k^2}}(f, \Psi_a) =$$

$$\sum_{i=a^{(k-1)^2}}^{a^{(k-1)^2}+m_k-1} C_i^{(k)} \cdot \psi_i(x) = \frac{1}{k^2} \sum_{i=a^{(k-1)^2}}^{a^{(k-1)^2}+m_k-1} \psi_i(x) +$$

$$\sum_{i=a^{(k-1)^2}}^{a^{(k-1)^2}+m_k-1} \frac{1}{2^i} \cdot \psi_i(x) = J_1 + J_2. \tag{29}$$

Taking into account (6) and (7) we get

$$J_1 = \frac{1}{k^2} \sum_{i=a^{(k-1)^2}}^{a^{(k-1)^2}+m_k-1} \psi_i(x) = \frac{1}{k^2} \sum_{i=0}^{m_k-1} \psi_{a^{(k-1)^2}+i}(x) =$$

$$\frac{1}{k^2} \cdot \psi_{a^{(k-1)^2}}(x) \cdot \sum_{i=0}^{m_k-1} \psi_i(x) = \frac{1}{k^2} \cdot \psi_{a^{(k-1)^2}}(x) \cdot D_{m_k}(x).$$

$$| J_2 | \le \sum_{i=a^{(k-1)^2}}^{a^{(k-1)^2}+m_k-1} \frac{1}{2^i} | \psi_i(x) | \le \sum_{i=a^{(k-1)^2}}^{\infty} \frac{1}{2^i} \le 2^{-a^{(k-1)^2}+1}.$$

From this and (29) we obtain

$$| G_{a^{(k-1)^2}+m_k}(f,\Psi_a) - G_{a^{k^2}}(f,\Psi_a) | \geq$$
$$\frac{1}{k^2}\cdot | \psi_{a^{(k-1)^2}}(x) || D_{m_k}(x) | -2^{-a^{(k-1)^2}+1} =$$
$$\frac{1}{k^2}\cdot | D_{m_k}(x) | -2^{-a^{(k-1)^2}+1}. \tag{30}$$

Now we take the sequence of natural numbers m_ν defined by Lemma (see (11) (12))such that $a^{(k-1)^2} \leq m_\nu < a^{(k-1)^2+1}$.

Then from (30) we have

$$\int_0^1 | G_{a^{(k-1)^2}+m_k}(f,\Psi_a) - G_{a^{k^2}}(f,\Psi_a) | \, dx >$$
$$\frac{1}{k^2}\cdot \int_0^1 |D_{m_k}(x)|dx - 2^{-a^{(k-1)^2}+1} \geq \frac{1}{2a\cdot k^2}\cdot \log_a m_k - 2^{-a^{(k-1)^2}+1} \geq$$
$$\frac{(k-1)^2}{2a\cdot k^2} - 2^{-a^{(k-1)^2}+1} \geq \frac{1}{4a} - 2^{-a^{(k-1)^2}+1} \geq C_1, \quad k \geq 4.$$

Thus the sequence $\{G_n(f,\Psi_a)\}$ does not converge by $L^1[0,1]$ norm, i.e. the generalized Walsh system Ψ_a is not a quasi-greedy basis in its linear span in $L^1[0,1]$.

Completing the proof.

R E F E R E N C E S

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